Closing Tues: 13.2, 13.3

Closing Thur: 13.4

Exam 1 is Thurs (April 19)

covers 12.1-12.6, 13.1-13.4

13.1-13.4 Curves in 3D

Given $\vec{\boldsymbol{r}}(t) = \langle x(t), y(t), z(t)\rangle$

$\vec{\boldsymbol{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ tangent vector

$$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(u)|du = \text{distance}$$

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3} = \text{curvature}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin(t) & 2\cos(t) & 0 \\ -2\cos(t) & -2\sin(t) & 0 \end{vmatrix} = \langle 0, 0, 4\sin^2(t) + 4\cos^2(t)\rangle$$
$$= \langle 0, 0, 4\rangle$$

Entry Task:

$$\vec{\boldsymbol{r}}(t) = \langle 2\cos(t), 2\sin(t), 0\rangle$$

Find $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, and K.

$\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 0\rangle$

$|\vec{r}'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = \sqrt{4} = 2$

$\vec{T}(t) = \frac{1}{2}\langle -2\sin(t), 2\cos(t), 0\rangle$

$\boxed{\vec{T}(t) = \langle -\sin(t), \cos(t), 0\rangle}$

ORTHOGONAL

$\vec{T}'(t) = \langle -\cos(t), -\sin(t), 0\rangle$

$|\vec{T}'(t)| = \sqrt{(-\cos(t))^2 + (-\sin(t))^2 + 0^2} = \sqrt{1} = 1$

$\vec{N}(t) = \frac{1}{1}\langle -\cos(t), -\sin(t), 0\rangle$

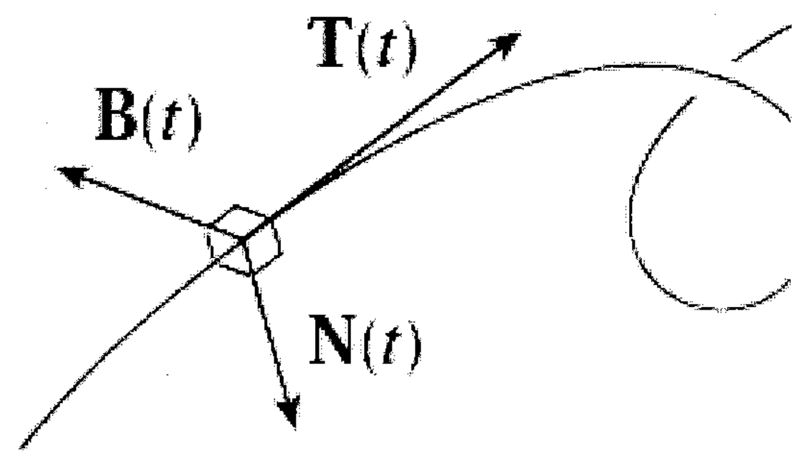
$\boxed{\vec{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle}$

$\vec{r}''(t) = \langle -2\cos(t), -2\sin(t), 0\rangle$

$$\boxed{\frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{\sqrt{0^2+0^2+4^2}}{2^3} = \frac{4}{8} = \frac{1}{2}}$$

Note:

$\boldsymbol{T}$ and $\boldsymbol{T}'$ are always orthogonal.



Proof:

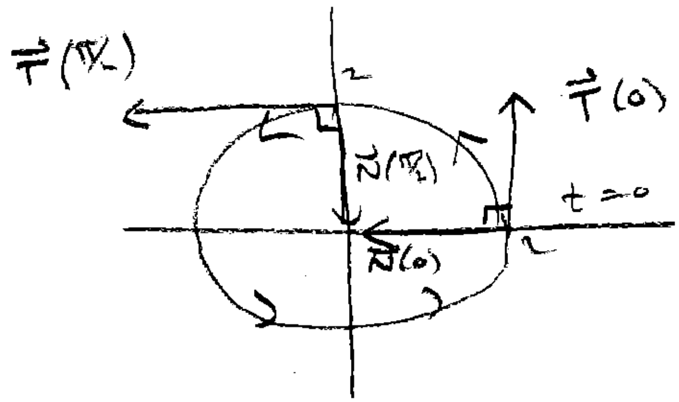
Since $\boldsymbol{T}\cdot\boldsymbol{T} = |\boldsymbol{T}|^2 = 1$, we can differentiate both sides to get

$$\boldsymbol{T}'\cdot\boldsymbol{T} + \boldsymbol{T}\cdot\boldsymbol{T}' = 0.$$

So $2\boldsymbol{T}\cdot\boldsymbol{T}' = 0$.

Thus, $\boldsymbol{T}\cdot\boldsymbol{T}' = 0$. (QED)

LAST EXAMPLE



$\vec T(0) = \langle 0, 1, 0\rangle$

$\vec N(0) = \langle -1, 0, 0\rangle$

$\vec T(\pi/2) = \langle -1, 0, 0\rangle$

$\vec N(\pi/2) = \langle 0, -1, 0\rangle$

$k = 1/2$

"RADIUS OF CURVATURE"

$= \dfrac{1}{k} = 2$

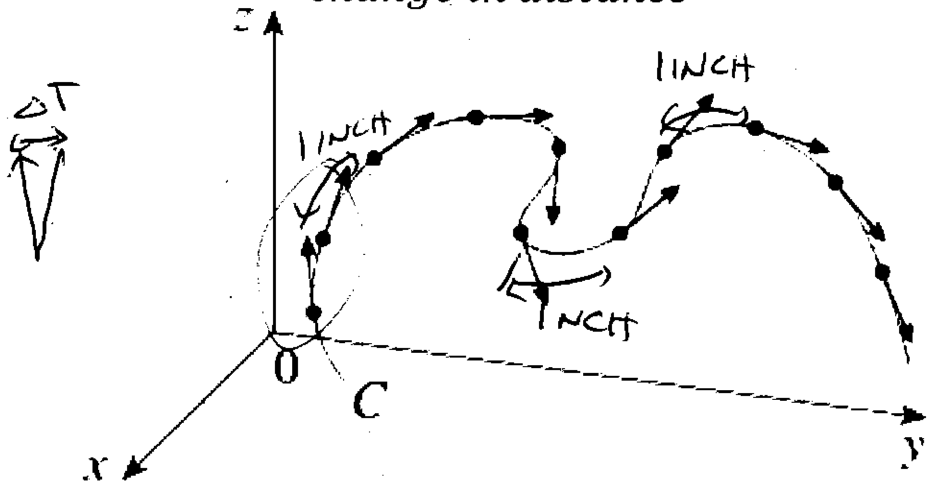
IT IS!

"CURVING LIKE A CIRCLE OF RADIUS 2"

Curvature

The **curvature** at a point, K, is a measure of how quickly a curve is changing direction at that point.

$$K = \frac{\text{change in direction}}{\text{change in distance}}$$



Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$K \approx \left|\frac{\vec{T_2}-\vec{T_1}}{\text{"one inch"}}\right| = \left|\frac{\Delta\vec{T}}{\Delta s}\right|$$

So we define:

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right|.$$

Computation Notes

(see my 13.3 Notes/book for the proof)

1st shortcut:

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \left|\frac{d\vec{\boldsymbol{T}}/dt}{ds/dt}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|}$$

2nd shortcut

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3}$$

Aside:

The *radius of curvature* is the radius of the circle that would best fit the curve at the given point.

$$\textit{radius of curvature} = \frac{1}{K}$$

2D Curvature

To find curvature for, $y = f(x)$,
We form the 3D vector function

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle.$$

so $\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$ and

$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\boldsymbol{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$

Thus,

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Example: $f(x) = x^2$
At what point (x, y, z) is the curvature maximum?

$f'(x) = 2x$

$f''(x) = 2$

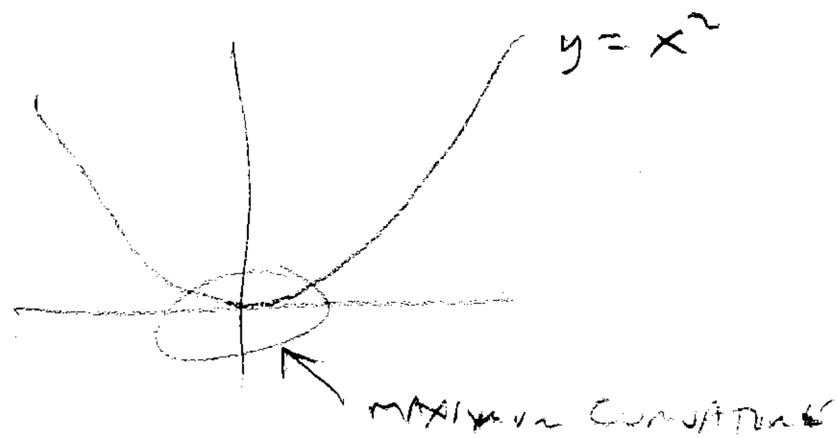
$$\boxed{K(x) = \frac{2}{\left(1 + (2x)^2\right)^{3/2}}} \leftarrow \text{MAXIMIZE!!!}$$

$K(x) = 2(1+4x^2)^{-3/2}$

$K'(x) = -3(1+4x^2)^{-5/2} \cdot 8x \overset{?}{=} 0$

$x = 0 \leftarrow$ critical #

$K(0) = 2 =$ maximum



13.4 Position, Velocity, Acceleration

If *t = time* and position is given by

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$$

then

$$\boldsymbol{r}'(t) = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h)-\boldsymbol{r}(t)}{h}$$
$$= \frac{\text{change in position}}{\text{change in time}}$$
$$= \text{velocity} = \boldsymbol{v}(t)$$

$$|\boldsymbol{r}'(\boldsymbol{t})| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed}$$

$$\boldsymbol{r}''(t) = \lim_{h\to 0}\frac{\boldsymbol{r}'(t+h)-\boldsymbol{r}'(t)}{h}$$
$$= \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \text{acceleration} = \boldsymbol{a}(t)$$

Let t be **time in seconds** and assume the position of an object (in **feet**) is given by

$$\boldsymbol{r}(t) = < t, 2e^{-t}, 0 >$$

Compute

1. $\boldsymbol{r}'(t)$, $|\boldsymbol{r}'(t)|$, and $\boldsymbol{r}''(t)$.
2. $\boldsymbol{r}'(0)$, $|\boldsymbol{r}'(0)|$, and $\boldsymbol{r}''(0)$.

$\vec{r}\,'(t) = \langle 1, -2e^{-t}, 0\rangle$

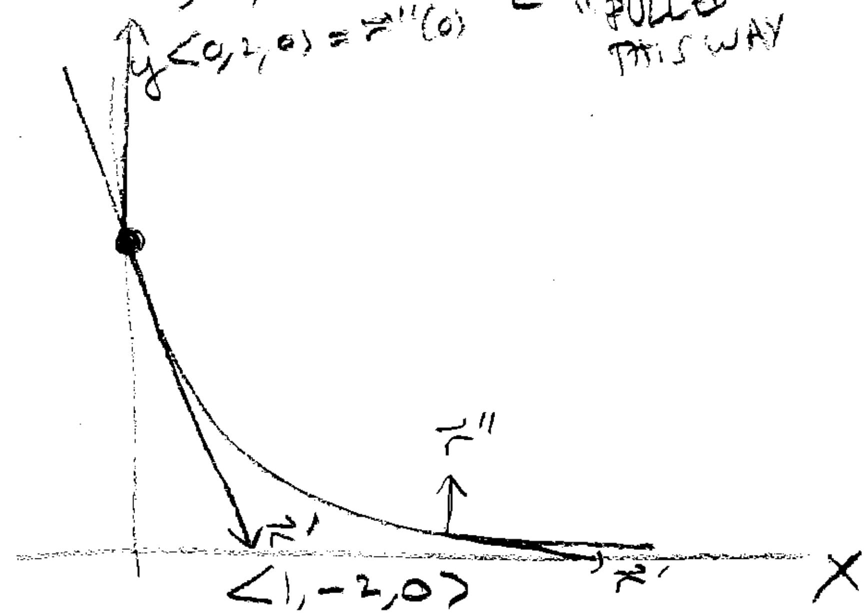
$|\vec{r}\,'(t)| = \sqrt{1+4e^{-2t}}$

$\vec{r}\,''(t) = \langle 0, 2e^{-t}, 0\rangle$

$\vec{r}\,'(0) = \langle 1, -2, 0\rangle$

$|\vec{r}\,'(0)| = \sqrt{5}$

$\vec{r}\,''(0) = \langle 0, 2, 0\rangle$ ← BEING "PULLED" THIS WAY



HUGE application:
Modeling ANY motion problem.

Newton's 2nd Law of Motion states

Force = mass · acceleration

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \text{ , so}$$

$$\boldsymbol{a} = \frac{1}{m} \cdot \boldsymbol{F}$$

If $\boldsymbol{F} = \langle 0,0,0 \rangle$, then all the forces 'balance out' and the object has no acceleration. (Velocity will remain constant)

If $\boldsymbol{F} \neq \langle 0,0,0 \rangle$, then acceleration will occur, and we integrate (or solve a differential equation) to find velocity and position.

That is how we can model ALL motion problems!

HW Example:

An object of mass 10 kg is being acted on by the force $\boldsymbol{F} = \langle 130t, 10e^{t}, 10e^{-t} \rangle$.

You are given

$$\boldsymbol{v}(0) = \langle 0,0,1 \rangle \text{ and } \boldsymbol{r}(0) = \langle 0,1,1 \rangle.$$

Find the position function.

$\vec{a}(t) = \frac{1}{m}\vec{F} = \langle 13t, e^{t}, e^{-t} \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt$

$= \langle \frac{13}{2}t^2 + c_1, e^{t} + c_2, -e^{-t} + c_3 \rangle$

$\vec{v}(0) = \langle 0,0,1 \rangle \Rightarrow \langle c_1, 1 + c_2, -1 + c_3 \rangle = \langle 0,0,1 \rangle$

$c_1 = 0, \quad c_2 = -1, \quad c_3 = 2$

$\vec{v}(t) = \langle \frac{13}{2}t^2, e^{t} - 1, -e^{-t} + 2 \rangle$

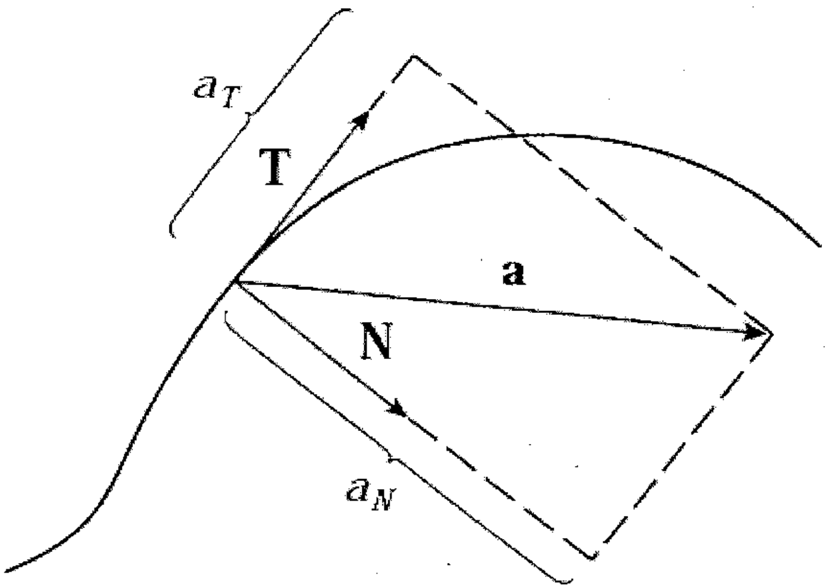
$\vec{r}(t) = \langle \frac{13}{6}t^3 + d_1, e^{t} - t + d_2, e^{-t} + 2t + d_3 \rangle$

$\vec{r}(0) = \langle 0,1,1 \rangle \Rightarrow \langle d_1, 1 + d_2, 1 + d_3 \rangle = \langle 0,1,1 \rangle$

$d_1 = 0, \ d_2 = 0, \ d_3 = 0$

$$\boxed{\vec{r}(t) = \langle \frac{13}{6}t^3, e^{t} - t, e^{-t} + 2t \rangle}$$

Measuring and describing acceleration



Recall: $\text{comp}_{\boldsymbol{b}}(\boldsymbol{a}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}}$ = length.

We define the tangential and normal components of acceleration by:

$a_T = \text{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{T}$ = tangential

$a_N = \text{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{N}$ = normal

For computing use,

$$a_T = \frac{\vec{\boldsymbol{r}}'\cdot\vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|} \quad \text{and} \quad a_T = \frac{|\vec{\boldsymbol{r}}'\times\vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|}$$

For interpreting use,

$a_T = v' = \frac{d}{dt}|r'(t)|$ = "deriv. of speed"

$a_N = kv^2$ = curvature · (speed)^2

Example:

$$\vec{r}(t) = < \cos(t)\,, \sin(t)\,, t >$$

Find the tangential and normal components of acceleration.

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0\rangle$$

$$\vec{r}\,' \cdot \vec{r}\,'' = \cos(t)\sin(t) - \cos(t)\sin(t) + 0 = 0$$

$$\Rightarrow \quad a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} = 0 \Leftrightarrow \text{CONSTANT SPEED}$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = (0 - -\sin(t))\vec{i} - (0 - -\cos(t))\vec{j} + (\sin^2(t) - -\cos^2(t))\vec{k}$$

$$= \langle \sin(t), -\cos(t), 1\rangle \qquad \text{Check} \checkmark$$

$$|\vec{r}\,' \times \vec{r}\,''| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$a_N = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|} = \frac{\sqrt{2}}{\sqrt{2}} = 1 = K\,(\overset{\sqrt{2}}{\overset{\downarrow}{\text{speed}}})^2 \Rightarrow K = 1/2$$

RADIUS OF CURVATURE = 2

Deriving interpretations:

Note that: $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$

Let $\nu(t) = |\vec{\boldsymbol{v}}(t)|$ = speed.

1. $\vec{\boldsymbol{T}}(t) = \dfrac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \dfrac{\vec{\boldsymbol{v}}(t)}{\nu(t)}$ implies $\vec{\boldsymbol{v}} = \nu\vec{\boldsymbol{T}}$.
2. $\kappa(t) = \dfrac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \dfrac{|\vec{\boldsymbol{T}}'|}{\nu(t)}$ implies $|\vec{\boldsymbol{T}}'| = \kappa\nu$.
3. $\vec{\boldsymbol{N}}(t) = \dfrac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \dfrac{\vec{\boldsymbol{T}}'}{\kappa\nu}$ implies $\vec{\boldsymbol{T}}' = \kappa\nu\vec{\boldsymbol{N}}$.

Differentiating the first fact above gives

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + \nu\vec{\boldsymbol{T}}', \text{ so}$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + k\nu^2\vec{\boldsymbol{N}}.$$

Conclusion:

$a_T = \nu' = \frac{d}{dt}|r'(t)|$ = "deriv. of speed"

$a_N = k\nu^2$ = curvature · (speed)^2